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A linear connection for higher-order ordinary differential equations

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Abstract. A linear connection is defined on the space $J^N E$ of N-jets of sections of $\tau : E \to B$, where B is one-dimensional. This is a first step toward classifying $(N+1)$th-order, time-dependent ordinary differential equations. The module of vector fields on $J^N E$ splits into coordinate invariant components, $N+1$ of them isomorphic and one component generated by the differential equation field. The connection is shown to uniquely determine the differential equation field to which it is associated.

1. Introduction

Recent advances in the geometric theory of second-order ordinary differential equations (SODE) have provided a linear algebraic criterion for the complete decoupling of systems of SODE [2, 13] and significant progress in the inverse problem of Lagrangian mechanics [7]. Within the last two years, there have been results which make it possible to identify those systems of SODE which are equivalent to trivial systems [5] or to systems in which the forces are independent of velocities [14]. Central to these results have been two geometric objects, the *dynamical covariant derivative* ∇ and the *Jacobi endomorphism* Φ, defined in [12, 13]. These objects arise in the theory of tensor fields defined along the tangent bundle projection, developed principally by Martínez, Cariñena and Sarlet [11, 12, 19]. They are a development of the generalized vector fields originally introduced by Johnson [9].

Given the utility of these objects in the study of SODE, it is reasonable to ask if similar structures can be found for systems of higher-order ordinary differential equations (HODE). This paper is a first step towards their construction.

Some of the necessary structure is known: a paper published in 1986 by Crampin, Sarlet and Cantrijn [6] explored the geometry of HODE on the higher-order tangent bundle $T^N M$ (in the case of $(N+1)$th-order ODE on the manifold M). This built on earlier work by authors such as Tulczyjew [20], de Léon [8] and Krupka and Musilova [10]. In particular Crampin *et al* established conditions for a given system of HODE to be derivable from a higher-order Lagrangian and formulated a 'prototype Noether's theorem' which reduced to Noether's in the SODE case. There has also been recent work directed at applying the theory of vector fields along the projection to HODE [3]. However, the core of the theory of vector fields along the tangent bundle projection is that the module $\chi(TM)$ of vector fields on TM splits as the direct sum of *vertical* and *horizontal* submodules,

$$\chi(TM) = \mathcal{V}(TM) \oplus \mathcal{H}(TM)$$

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with the $\mathcal{V}(TM)$ and $\mathcal{H}(TM)$ being isomorphic. The choice of a horizontal submodule specifies a horizontal subspace of $T_p(TM)$ at each point $p \in TM$ and is therefore equivalent to a Cartan–Ehresmann connection on TM (see for example [4, 15]), which can in turn be used to define a linear connection on the jet space J^1E [1]. I will show in section 3 that a similar linear connection can be constructed on J^NE provided there is an appropriate invariant decompostion of $\chi(J^NE)$.

In the SODE theory this connection is determined by $\mathcal{L}_\Gamma S$, the Lie derivative of the vertical endomorphism with respect to the differential equation field. Hence the problem with the current state of knowledge of higher-order systems: the eigenspaces of $\mathcal{L}_\Gamma S$ are too big. Although Crampin *et al* in [6] use the HODE to define a splitting

$$\chi(T^NM) = \mathcal{V}(TM) \oplus \mathcal{H}(TM)$$

the dimension of $\mathcal{H}(TM)$ is N times that of $\mathcal{V}(TM)$, so they cannot be isomorphic when $N > 1$. In their recent work Cariñena, Lopez and Martínez do not address the question of splitting. Since they study the higher-order tangent bundle as a nested structure of tangent bundles, they are confronted by the lack of a differential equation field with which to define a connection except on the 'top' tangent space.

What is really needed is a further splitting of the horizontal space

$$\chi(T^NM) = \mathcal{V}(T^NM) \oplus \bigoplus_{m=0}^{N-1} \mathcal{H}^{(m)}(T^NM)$$

such that all the submodules $\mathcal{H}^{(m)}(TM)$ are isomorphic to $\mathcal{V}(TM)$. Of course they should also be invariant under coordinate transformations, disqualifying a naive approach based on induced coordinates.

In this paper I will prove that such a splitting exists and explicitly construct the submodules $\mathcal{H}^{(m)}$ and the linking isomorphisms in section 4. In fact I will work not with T^NM but in the more general setting of the Nth-order jet bundle J^NE, where $\tau : E \to B$ is a fibre bundle over a one-dimensional base (the independent variable) whose fibres are homeomorphic to M. This allows the study of time-dependent HODE ('time' being a generic term for the independent variable).

In section 5 I note some properties of the resulting linear connection. In particular it will be proved that the map from HODE to linear connection is one-to-one. Hence the classification of HODE up to time-dependent transformations of the dependent variables is equivalent to the classification of the associated linear connections.

2. Background and notation

Let $\tau : E \to B$ be a smooth fibre bundle with one-dimensional base B. I will write J^NE for the Nth-order jet-bundle of sections $B \to E$. A typical fibre $\tau^{-1}(b)$ will be called M. Then if t is a local coordinate on B and $x \equiv (x^1, \ldots, x^m)$ are local coordinates on M, I will implicitly assume some local trivialization and take $(t, x) \equiv (t, x^1, \ldots, x^m)$ to be local coordinates on E. These induce coordinates $(t, x, x_{(1)}, \ldots, x_{(N)})$ for J^NE, so that the point with these coordinates is the N-jet containing the section

$$s \mapsto x + x_{(1)}(s-t) + \tfrac{1}{2}x_{(2)}(s-t)^2 + \cdots + \frac{1}{N!}x_{(N)}(s-t)^N.$$

Using this correspondence, smooth coordinate transition functions on E induce smooth transition functions on J^NE. I will also use the standard notation $f(t, [x])$ for a function evaluated at some point of J^NE, with $[x]$ denoting dependence on the derivatives of x.

The prolongation of a section is defined by the same correspondence. If $\sigma : B \to E$ is a section, then the Taylor series expansion of σ about t determines a point $\sigma_{(N)}(t)$ in $(\tau \circ \pi_N^0)^{-1}(t)$. So if the coordinate form of σ is $t \mapsto \sigma(t)$, then the prolongation $\sigma_{(N)}$ has the local expression

$$\sigma_{(N)} : t \mapsto \left(t, \sigma(t), \frac{\mathrm{d}\sigma}{\mathrm{d}t}(t), \ldots, \frac{\mathrm{d}^N\sigma}{\mathrm{d}t^N}(t)\right).$$

The projection $\pi_N^n : J^N E \to J^n E$, $n < N$, is defined in coordinates by truncation. Thus there is a projection $\pi_N^0 : J^N E \to E$ by the identification $J^0 \equiv E$.

The module of sections of the tangent bundle $TJ^N E$, denoted $\chi(J^N E)$, is spanned by the vector fields

$$\left\{\frac{\partial}{\partial t}, \frac{\partial}{\partial x^a}, \frac{\partial}{\partial^a_{(1)}}, \ldots, \frac{\partial}{\partial x^a_{(N)}} : a = 1, \ldots, m\right\}.$$

For the sake of convenience and clarity, from here on I will write $X_a^{(j)} := \partial/\partial x^a_{(j)}$. It should be noted that for $j < N$, $X_a^{(j)}$ is not a point-wise function of $\partial/\partial x^a$. For this reason I will insist on distinguishing $X_a^{(0)}$, a vector field on $J^N E$, from $X_a := \partial/\partial x^a$ as a vector field on E. They are related by the projection $\pi^0_{N*} X_a^{(0)} = X_a$.

It will be convenient to define the module

$$\bar{\chi}(J^N E) := \{Y \in \chi(J^N E) : Y(t) = 0\}$$

of vector fields with no time-like component.

If Y is vector field on E, then the flow ψ_ϵ of Y carries sections of τ to sections by deforming their graphs. Note that such flows need not preserve the fibres of τ: see [16]. Since sections define points in $J^N E$, there is a prolongation of ψ_ϵ to a flow $\psi_\epsilon^{(N)}$ of $J^N E$. This allows us to define the prolongation Y^C of Y by requiring

$$Y^C(f) = \frac{\mathrm{d}}{\mathrm{d}\epsilon} f \circ \psi_\epsilon^{(N)}$$

to hold for all smooth $f : J^N E \to \mathbb{R}$. The C stands for *complete lift*, a term used by some authors [15] in place of prolongation. The coordinate form of Y^C, if $Y = Y^0 \frac{\partial}{\partial t} + Y^a X_a \in \chi(E)$, is

$$Y^C = Y^0 \frac{\partial}{\partial t} + Y^a X_a^{(0)} + Y^a_{(1)} X_a^{(1)} + \cdots + Y^a_{(N)} X_a^{(N)}$$

where

$$Y^a_{(k)} := \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^k (Y^a - x^a_{(1)} Y^0) \qquad k = 1, \ldots, N.$$

The contact forms on $J^N E$ are given by

$$\theta^a_{(j)} := dx^a_{(j)} - x^a_{(j+1)} \mathrm{d}t \qquad j = 0, \ldots, N-1.$$

The ideal generated by the contact forms will be denoted Θ: while the contact forms are not tensorial with respect to coordinate changes on E, the ideal Θ is invariant under such changes. Suppose that $\sigma : B \to E$ is a section whose prolongation gives a section $\sigma_{(N)} : B \to J^N E$. Then the pull-back of the contact forms satisfies $\sigma^*_{(N)} \Theta = 0$. Conversely, any section of $\tau \circ \pi_N^0 : J^N E \to B$ which annihilates the contact forms in this way is the prolongation of a section of $\tau : E \to B$.

The vertical endomorphism S can be defined in terms of the contact forms as

$$S := \sum_{j=0}^{N-1}(j+1)\theta^a_{(j)} \otimes X_a^{(j+1)}.$$

Note that

$$S(X_a^{(j)}) = (j+1)X_a^{(j+1)} \qquad j = 0, \ldots, N-1 \qquad S(X_a^{(N)}) = 0.$$

Using the fact that

$$S\left(\frac{\partial}{\partial t}\right) = -\sum_{j=1}^{N} j x^a_{(j)} X_a^{(j)}$$

the Nijhenhuis bracket $\mathcal{N}_S$ of S can be shown to satisfy

$$\tfrac{1}{2}\mathcal{N}_S(Y, Z) = Y(t)S(Z) - Z(t)S(Y).$$

Thus the result $\mathcal{L}S(Y)S = S \circ \mathcal{L}YS$ derived in [6] for the autonomous case must be modified:

$$(\mathcal{L}S(Y)S)(Z) - S((\mathcal{L}YS)(Z)) = Y(t)S(Z) - Z(t)S(Y). \tag{1}$$

2.1. *The HODE field*

Definition 1. The vector field $\Gamma \in \chi(J^N E)$ is an $(N+1)$th-order ODE field, or more generally a higher-order ODE field (HODE), if it satisfies the two conditions

$$\Gamma \in \Theta^{\perp} \qquad \Gamma(t) = 1.$$

The second of these conditions effectively fixes the coordinates on B: since B is one-dimensional, the resulting loss of generality is not great.

In coordinates, a HODE Γ will have the form

$$\Gamma(t, [x]) = \frac{\partial}{\partial t} + x^a_{(1)}X_a^{(0)} + \cdots + x^a_{(N)}X_a^{(N-1)} + f^a(t, [x])X_a^{(N)}.$$

It can easily be checked that the integral curves of Γ are prolongations of solutions of the system of ODE

$$\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{N+1} x^a = f^a(t, x, \ldots, x_{(N)}) \qquad a = 1, \ldots, m.$$

Moreover every solution of this system is the projection of an integral curve of Γ.

3. Linear connections

Consider a manifold M with a group of automorphisms G. There is an induced action of G on the module of vector fields $\chi(M)$. Suppose $\chi(M)$ has a G-invariant decomposition

$$\chi(M) = \langle\xi\rangle \oplus A_0 \oplus \cdots \oplus A_N \qquad N > 1$$

where $\langle\xi\rangle$ is the one-dimensional submodule generated by ξ. I will require that ξ is invariant under G, in addition to the submodule it generates. Assume also that there exist $C^\infty(M)$-isomorphisms $\sigma_{j,k} : A_j \to A_k$ for all $0 \leqslant j, k \leqslant N$, with $\sigma_{j,k} \circ \sigma_{k,j} = \mathrm{id}_j$. Let $P_{A(k)}$ be the projection $\chi(M) \to A_k$ and P_ξ the projection $\chi(M) \to \langle\xi\rangle$.

Now define the $\mathbb{R}$-bilinear operator $\nabla : \chi(M) \times \chi(M) \to \chi(M)$ as follows. For all $Y_j \in A_j$ and $f, h \in C^\infty(M)$, set

$$\begin{aligned}
\nabla_{Y_j} Y_k &:= \sigma_{j+_N 1,k} \circ P_{A(j+_N 1)}([Y_j, \sigma_{k,j+_N 1}(Y_k)]) \\
\nabla_{f\xi} Y_j &:= P_{A(j)}([f\xi, Y_k]) \\
\nabla_{Y_j}(f\xi) &:= \nabla_{f\xi} Y_j + [Y_j, f\xi] \\
\nabla_{f\xi}(h\xi) &:= f\xi(h)\xi.
\end{aligned}$$

The addition $+_N$ in the first of these equations is to be read as addition mod N.

Proposition 1. Defined as above, ∇ is a linear connection covariant with respect to G, so if $g \in G$ and its action on Y is $g \cdot Y$,

$$\begin{aligned}
\nabla_{fY} Z &= f \nabla_Y Z \\
\nabla_Y (fZ) &= Y(f)Z + f\nabla_Y Z \\
\nabla_{g\cdot Y}(g \cdot Z) &= g \cdot (\nabla_Y Z).
\end{aligned}$$

Proof. The linearity and Leibnitz properties can be checked directly. The G-covariance follows from the invariance of ξ, A_j and the covariance of the Lie bracket. □

In the next section I will show there is a decomposition of the jet-bundle $J^N E$ with the above properties, G being the group of automorphisms induced from the bundle automorphisms of $\tau : E \to B$. In other words, the decomposition is invariant under time-dependent transformations of the dependent variables. For the sake of brevity, I will refer to invariance (or covariance) with respect to this group of automorphisms as coordinate invariance (covariance). It should be understood that this is different from invariance with respect to transformation of $J^N E$ which are not induced from bundle automorphisms of $\tau : E \to B$.

To see how this works, recall the theory of second-order ODE (SODE)

$$\ddot{x}^a = f^a(t, x, \dot{x})$$

as found in [1, 5, 13]. Here the SODE field Γ is a vector field on the first jet-bundle $J^1 E$,

$$\Gamma = \frac{\partial}{\partial t} + x^a_{(1)} X^{(0)}_a + f^a(t, x, x_{(1)}) X^{(1)}_a.$$

The role of ξ in proposition 1 is played by Γ. To obtain the remainder of the decomposition, one calculates the Lie derivative $\mathcal{L}_\Gamma S$. It is easily shown that $\mathcal{L}_\Gamma S$ has three eigenvalues: $-1, 0, +1$. The eigenspace corresponding to the zero eigenvalue is spanned by Γ. The $+1$-eigenspace $\mathcal{V}$ is spanned by the π^0_1-vertical vector fields $V_a := X^{(1)}_a$, which can be constructed as $S(X^C_a)$. The remaining eigenspace $\mathcal{H}$ is called the horizontal space and is spanned by the (in general non-commutative) basis vectors

$$H_a := X^{(0)}_a - \Gamma^b_a V_b, \Gamma^b_a := -\tfrac{1}{2} V_a f^b.$$

From the definition, this decomposition is invariant under changes of coordinate. The vertical endomorphism restricts to the isomorphism $S : \mathcal{H} \to \mathcal{V}$, $S(H_a) = V_a$. Its inverse I will denote σ. Following proposition 1, there is a linear connection $\bar{\nabla}$ defined by

$$\begin{aligned}
\bar{\nabla}_{H_a} H_b &= \sigma \circ P_V([H_a, V_b]) = \Gamma^c_{ab} H_c \\
\bar{\nabla}_{H_a} V_b &= P_V([H_a, V_b]) = \Gamma^c_{ab} V_c \\
\bar{\nabla}_{V_a} H_b &== \sigma \circ P_H([V_a, H_b]) = 0
\end{aligned}$$

$$\bar{\nabla}_{V_a} V_b == P_H([V_a, H_b]) = 0$$
$$\bar{\nabla}_\Gamma H_a = P_H([\Gamma, H_a]) = \Gamma_a^b H_b$$
$$\bar{\nabla}_\Gamma V_a = P_V([\Gamma, V_a]) = \Gamma_a^b V_b$$
$$\bar{\nabla}_{H_a} \Gamma = \bar{\nabla}_\Gamma H_a + [H_a, \Gamma] = -\Phi_a^b V_b$$
$$\bar{\nabla}_{V_a} \Gamma = \bar{\nabla}_\Gamma V_a + [V_a, \Gamma] = H_a$$
$$\bar{\nabla}_\Gamma \Gamma = 0.$$

Here Φ is a tensor defined in [13], with coordinate expression

$$\Phi_a^b := -X_a^{(0)}(f^b) - \Gamma_a^c \Gamma_c^b - \Gamma(\Gamma_a^b)$$

and $\Gamma_{ab}^c := V_a(\Gamma_b^c)$. The connection $\bar{\nabla}$ was first given in [1].

It should be noted that the above expressions are somewhat redundant, since $(\bar{\nabla}_Y S)(Z) = 0$ for all $Y \in \chi(J^1E)$ and all $Z \in \bar{\chi}(J^1E)$. This fact leads to a more concise notation developed by Martínez *et al* [11–13].

4. Invariant decomposition of $J^N E$

The differential equation field Γ and the vertical endomorphism S are defined for HODE as for SODE. Therefore the first trial one conducts is to calculate the eigenvalues of the linear map $\mathcal{L}_\Gamma S$. Unfortunately there are only three distinct eigenvalues, regardless of the value of N: these are $-1, 0$ and N. As in the $N = 1$ case, the 0-eigenspace is spanned by the ODE field Γ and the N-eigenspace consists of the *very vertical* vector fields

$$\mathcal{V} := \{V \in \bar{\chi}(J^N E) : \pi_{N*}^{N-1} V = 0\}.$$

As in the $N = 1$ case, there is a natural basis $\{V_1, \ldots, V_m\}$ for $\mathcal{V}$, with $V_a := X_a^{(N)} = S^N(X_a^C)$. It will be convenient to call the complementary -1-eigenspace the horizontal submodule $\mathcal{H}$, although it contains elements which are vertical with respect to π_N^0. The corresponding projections are

$$P_H := \frac{1}{N+1} \mathcal{L}_\Gamma S \circ (\mathcal{L}_\Gamma S - NI)$$
$$P_V := \frac{1}{N(N+1)} \mathcal{L}_\Gamma S \circ (\mathcal{L}_\Gamma S + I)$$
$$P_\Gamma := \mathrm{d}t \otimes \Gamma.$$

The three submodules $\langle\Gamma\rangle$, $\mathcal{V}$ and $\mathcal{H}$ are coordinate invariant, but $\dim(\mathcal{H}) = N \dim(\mathcal{V})$. However, $\mathcal{H}$ can be further decomposed using the following fact. When restricted to $\bar{\chi}(J^N E)$, $\ker S = \mathcal{V}$. Consequently S has a trivial kernel when restricted to $\mathcal{H}$, so we can define

$$\mathcal{H}^{N-1} := S^{-1}(\mathcal{V}) \cap \mathcal{H}.$$

The rest of the decomposition is defined inductively:

$$\mathcal{H}^{j-1} := S^{-1}(\mathcal{H}^j) \cap \mathcal{H} \qquad 1 \leqslant j \leqslant N-1.$$

For notational convenience, set $\mathcal{H}^N := \mathcal{V}$.

Theorem 2. With submodules $\mathcal{H}^j$ defined as above,

$$\chi(J^N E) \simeq \langle\Gamma\rangle \oplus \mathcal{H}^0 \oplus \cdots \oplus \mathcal{H}^N.$$

Moreover the submodules $\mathcal{H}^j$ are isomorphic for $j = 0, \ldots, N-1$ and the decomposition is coordinate invariant.

Proof. Since S and Γ are coordinate invariant and the decomposition is constructed from them alone, the decomposition is invariant. The projections P_Γ and $P_H + P_V$ construct the splitting

$$\chi(J^N E) = \langle\Gamma\rangle \oplus \bar{\chi}(J^N E).$$

As noted already, S is an isomorphism when restricted to $\mathcal{H}$. Therefore it remains only to show that the $\mathcal{H}^j$ have trivial intersection and that they exhaust $\bar{\chi}(J^N E)$. Suppose that $Y \in \mathcal{H}^j \cap \mathcal{H}^k$, $j < k$. Then

$$S^{N-k}(Y) \in \mathcal{H}^{N+j-k} \cap \mathcal{V} \subset \mathcal{H} \cap \mathcal{V} = 0.$$

Now let $Y \in \bar{\chi}(J^N E)$ be non-zero, so that $Y = \sum_{a,j} Y^a_j X^{(j)}_a$. Let $j_0 \geqslant 0$ be such that $Y^a_j = 0$ if $j < j_0$. Then $S^{N-j_0}(Y) \in \mathcal{V}$, so $Y \in \mathcal{H}^{j_0} \oplus \cdots \oplus \mathcal{H}^N$. $\square$

For the isomorphism $\sigma_{j,j+1} : \mathcal{H}^j \to \mathcal{H}^{j+1}$ I will use

$$\sigma_{j,j+1} = \frac{1}{j+1} S \qquad 0 \leqslant j < N.$$

The remaining isomorphisms required by proposition 1 can be constructed from $\sigma_{j,j+1}$ in an obvious way. The benefit of this choice is that

$$\sigma_{j,N} : X^{(j)}_a \mapsto V_a \qquad 0 \leqslant j < N.$$

This can be used to give a coordinate form for the inverse map $\sigma_{j,j-1}$:

$$\sigma_{j,j-1} : X^{(j)}_a \mapsto P_H(X^{(j-1)}_a).$$

It can then be shown that the basis elements $H^{(j)}_a := \sigma_{N,j}(V_a)$ have the form

$$H^{(k)}_a = \frac{N!}{k!} \sigma^{N-k}(V_a) \tag{2}$$

$$= X^{(k)}_a - \sum_{j=0}^{N-k-1} \frac{(j+k+1)!(N-j)!}{N!\,k!} \Gamma^{(N-j)b}_a H^{(j+k+1)}_b \tag{3}$$

with the convention $H^{(N)}_a = V_a$.

The projections $P_{H(j)} : \chi(J^N E) \to \mathcal{H}^j$ can now be calculated in terms of the $H^{(j)}_a$:

$$P_{H(j)} = \left(\theta^a_{(j)} + \sum_{k=0}^{j-1} \theta^b_{(k)} \frac{j!(N-j+k+1)!}{N!k!} \Gamma^{(N-j+k+1)a}_b \right) \otimes H^{(j)}_a.$$

Here I have introduced the notation

$$\Gamma^{(k_1\cdots k_r)}a_{b_1\cdots b_r} := -\frac{1}{N+1} X^{(k_1)}_{b_1} \ldots X^{(k_r)}_{b_r}(f^a).$$

Corollary 3. There exists a linear connection on $J^N E$ which is coordinate covariant.

It would obviously be desirable to express the $H^{(j)}_a$ directly in terms of the coordinate basis $X^{(k)}_a$, however, for general N I have not been able to do this. Therefore I will conclude this section with some explicit formulae for $N = 2$ and $N = 3$ (the $N = 1$ case collapses to the well known expressions given in [15]).

For $N = 2$,

$$H^{(1)}_a = X^{(1)}_a - 2\Gamma^{(2)b}_a V_b$$
$$H^{(0)}_a = X^{(0)}_a - \Gamma^{(2)b}_a X^{(1)}_b + (2\Gamma^{(2)c}_a \Gamma^{(2)b}_c - \Gamma^{(1)b}_a) V_b.$$

For $N = 3$,

$$H_a^{(2)} = X_a^{(2)} - 3\Gamma_a^{(3)b} V_b$$
$$H_a^{(1)} = X_a^{(1)} - 2\Gamma_a^{(3)b} X_b^{(2)} + (6\Gamma_a^{(3)c}\Gamma_c^{(3)b} - 2\Gamma_a^{(2)b}) V_b$$
$$H_a^{(0)} = X_a^{(0)} - \Gamma_a^{(3)b} X_b^{(1)} + (2\Gamma_a^{(3)c}\Gamma_c^{(3)b} - \frac{2}{3}\Gamma_a^{(2)b}) X_b^{(2)}$$
$$+(2\Gamma_a^{(2)c}\Gamma_c^{(3)b} - 6\Gamma_a^{(3)d}\Gamma_d^{(3)c}\Gamma_c^{(3)b} - 2\Gamma_a^{(3)c}\Gamma_c^{(2)b} - \Gamma_a^{(1)} b) V_b.$$

5. Towards classification

Together with the decomposition of $\chi(J^N E)$ given in section 4, the linear connection defined in section 3 has several useful properties. The first, following immediately from the definition of $\bar{\nabla}$, is that it leaves the decomposition $\mathcal{H}^0 \oplus \cdots \oplus \mathcal{H}^N$ invariant.

Proposition 4. If $Y \in \chi(J^N E)$ and $Z \in \mathcal{H}^j$, $j = 0, \ldots, N$, then $\bar{\nabla}_Y Z \in \mathcal{H}^j$.

Also following directly from the definition is that the integral curves of Γ are geodesics (in the sense of being self-parallel: there is no guarantee that the connection $\bar{\nabla}$ preserves any metric on $J^N E$).

Proposition 5. If $\bar{\nabla}$ is the linear connection associated to the HODE field Γ, then $\bar{\nabla}_\Gamma \Gamma = 0$.

The above two results can be used to show that the connection encodes all of the information contained in the system of HODE, since the connection uniquely determines the HODE field Γ.

Theorem 6. If two HODE fields Γ and Γ' determine the same connection $\bar{\nabla}$, then $\Gamma' = \Gamma$.

Proof. Since Γ and Γ' must both annihilate the contact forms $\theta^a_{(j)}$ and satisfy $\Gamma(t) = \Gamma'(t) = 1$, there must exist $g^a \in C^\infty(J^N E)$ such that $\Gamma' = \Gamma + g^a V_a$. From proposition 5 we have that

$$\bar{\nabla}_\Gamma \Gamma = 0 = \bar{\nabla}_{\Gamma + g^a V_a}(\Gamma + g^a V_a)$$

implying that

$$g^a \bar{\nabla}_{V_a} \Gamma + \bar{\nabla}_\Gamma(g^a)\, V_a + g^a \bar{\nabla}_\Gamma V_a = 0. \tag{4}$$

The last two terms are contained in $\mathcal{V}$ by proposition 4, so equation (4) implies that

$$(g^a \bar{\nabla}_{V_a}\Gamma, dx^b_{(N-1)}) = 0.$$

Now $\bar{\nabla}_{V_a}\Gamma = \bar{\nabla}_\Gamma V_a - [\Gamma, V_a]$, so we must have

$$0 = g^a([\Gamma, V_a], dx^b_{(N-1)}) = -g^b x^b_{(N)} \text{ (no sum over } b).$$

Continuity then gives the result $g^a = 0$, $a = 1, \ldots, m$. □

As a consequence of this theorem, classification of systems of HODE up to time-dependent transformations of the dependent variables is equivalent to the classification of the connection $\bar{\nabla}$.

6. Discussion

The results in this paper are only a first step toward a coordinate invariant classification of systems of higher-order ordinary differential equations. Having discovered an appropriate linear connection, the immediate next step is to examine its curvature and torsion. While that could have been done in this paper, the properties of these objects lead to a rather large amount of new material.

In particular, the fact that the connection preserves the decomposition $\chi(J^N E) = \mathcal{H}^0 \oplus \cdots \oplus \mathcal{H}^N$ is reflected in a decomposition of the curvature tensor into a set of tensors defined along the projection $\pi_N^0 : J^N E \to E$. One can easily show that these curvature component tensors give necessary conditions for systems of HODE to be equivalent. In the $N = 1$ case they are the various objects Φ, θ, R and Rie defined in, for example, [13]. This was shown in [1].

To what extent such necessary conditions are also sufficient is still an open question. It is to be expected that there are results analogous to those in [5, 14] for the HODE case. In particular, it seems clear that vanishing curvature and torsion, together with a few other conditions, ought to imply equivalence to the trivial system $x^a_{(N+1)} = 0$.

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